## 3.3: Homogeneous Equations with Constant Coefficients

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We restrict our attention to equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

and define the **characteristic** (or **auxiliary**) equation by

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{2}$$

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**Theorem 1.** (Distinct Real Roots)
If the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation (2) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

is a general solution to (1).

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**Example 1.** Solve the initial value problem

$$y^{(3)} + 3y'' - 10y' = 0, \quad y(0) = 7, y'(0) = 0, y''(0) = 70.$$

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**Theorem 2.** (Repeated Roots)
If the characteristic equation (2) has a repeated real root $r$ of multiplicity $k$, then the part of a general solution of the differential equation (1) corresponding to $r$ is of the form

$$(c_1 + c_2 x + \cdots + c_{k-1} x^{k-2} + c^k x^{k-1}) e^{rx}.$$

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**Example 2.** Find a general solution to the equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0.$$

Next we wish to consider what happens if we do not have all real roots. In order to do this, we use Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3}$$

to give us a hint. Therefore

$$e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx).$$

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**Theorem 3.** (Complex Roots)
If the characteristic equation (2) has a complex root $r = a + ib$, then the part of a general solution of the differential equation (1) corresponding to $r$ is of the form

$$e^{ax}(c_1\cos bx + c_2\sin bx).$$

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**Exercise 1.** Show that $y(x) = c_1\cos bx + c_2\sin bx$ is a solution to

$$y'' + b^2y = 0.$$

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**Example 3.** Find the particular solution to the initial value problem

$$y'' - 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 5.$$

**Exercise 2.** Find the general solution to

$$y^{(4)} + 4y = 0.$$

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**Theorem 4.** (Repeated Complex Roots)
If the characteristic equation (2) has a repeated complex root $r = a + ib$ of multiplicity $k$, then the part of a general solution of the differential equation (1) corresponding to $r$ is of the form

$$\sum_{p=0}^{k-1} x^p e^{ax}(c_p \cos bx + c_p \sin bx).$$

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**Example 4.** Find the general solution to the differential equation whose characteristic equation has roots $3, -5, 0, 0, 0, 0, -5, 2 \pm 3i$ and $2 \pm 3i$.

**Exercise 3.** Find the general solution to the differential equation

$$y^{(3)} + y' - 10y = 0.$$

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**Homework.** 1-31 (odd)